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Supersymmetric Toda lattice hierarchy^{*}

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Abstract

The origin of the bosonic and fermionic solutions, constructed in [1, 2, 3], to the symmetry equations corresponding to the two-dimensional bosonic and $N = (2|2)$ supersymmetric Toda lattices is established, and algebras of the corresponding symmetries are derived. The conjecture regarding an $N = (2|2)$ superfield formulation of the $N = (2|2)$ supersymmetric Toda lattice hierarchy, proposed in [16], is proved. The two-dimensional $N = (0|2)$ supersymmetric Toda lattice hierarchy is proposed and its $N = (0|2)$ superfield formulation is discussed. Bosonic and fermionic solutions to the symmetry equation corresponding to the two-dimensional $N = (0|2)$ supersymmetric Toda lattice equation and their algebra are constructed. An infinite class of new two-dimensional supersymmetric Toda-type hierarchies is discussed.

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1 Introduction

Recently, an infinite class of solutions to the symmetry equation of the two-dimensional Toda lattice (2DTL) has been described in [1] in the framework of a rather heuristic algorithm of simple calculations proposed there. This algorithm resembles a computer program: It is necessary to perform many identical operations that can be interrupted at an arbitrary step and thus obtain relevant information about a system of $(2+1)$ -dimensional evolution equations belonging to the integrable 2DTL hierarchy. Then, this algorithm has been generalized to the case of the $N = (1|1)$ supersymmetric 2DTL equation, and an infinite class of bosonic solutions to its symmetry equation has been constructed in [2]. However, supersymmetry suggests that the symmetry equation possesses fermionic solutions as well, and they are responsible for fermionic flows of the hierarchy. Ref. [3] dwelled upon this problem and derived a wide class of fermionic solutions. Bosonic and fermionic solutions generate bosonic and fermionic flows of the $N = (1|1)$ supersymmetric 2DTL hierarchy in the same way as their bosonic counterparts – the solutions to the symmetry equation of the 2DTL – produce flows of the 2DTL hierarchy.

For a more complete understanding of an equation and its solutions it seems necessary to know as many solutions to the corresponding symmetry equation as possible. But a symmetry equation represents a complicated nonlinear functional equation, and its both general and particular solutions are not known in general. Moreover, in general there even exists no algorithm to solve this problem. As an illustration of the latter fact, let us mention, e.g. the unsolved yet problem of constructing symmetries to the $N = (0|2)$ supersymmetric 2DTL equation proposed in [4]. Due to this reason the algorithm developed in [1, 2, 3] and the resulting solutions to the symmetry equations corresponding to the 2DTL and $N = (1|1)$ 2DTL equations, as they have been presented so far, may appear to have come out of the blue, and it is interesting to understand their origin. In this connection, let us point out that long ago one of the authors of the present paper (V.G.K.), when analyzing an application of difference equations to solving problems of mathematical physics, developed an efficient approach to constructing solutions of some relativistic equations (for details, see refs. [5, 6, 7, 8, 9, 10] and references therein). Then this approach was successfully applied in [11, 12] to investigate the field theory with the momentum

space of a constant curvature where difference equations naturally arise in the configuration space and the lattice spacing is defined by the inverse radius of the curvature of the initial momentum space. Furthermore, it was recently adapted in [13] to the case of the gauge field theory. It turns out that this approach is also instructive in the context of the problem under consideration. Thus, our goal here is to establish the origin of the algorithm and symmetries of refs. [1, 2, 3] by reproducing them in the framework of the previously known integrable discrete hierarchies – the 2DTL [14] and super-Toda lattice (STL) [15] hierarchies – containing the 2DTL and $N = (1|1)$ 2DTL equations, respectively, as subsystems.

It is time to explain how we were led to this construction. Refs. [1, 2, 3, 5, 6, 7, 8, 9, 10] and [14, 15] can be considered ancestors of the present paper. As one might suspect, there is a correspondence between symmetries of the 2DTL and $N = (1|1)$ 2DTL equations and flows of the latter hierarchies, but this correspondence is however rather nontrivial. Thus, the 2DTL (STL) hierarchy has been defined in [14] ([15]) as a system of infinitely many equations for *infinitely many* fields, while the 2DTL ($N = (1|1)$ 2DTL) equation involves only a *single independent* (super)field $v_{0,j}$. From the point of view of the former approach the derivation of symmetries of the 2DTL ($N = (1|1)$ 2DTL) equation corresponds to extracting those 2DTL (STL) hierarchy equations which can be realized in terms of the (super)field $v_{0,j}$ alone after excluding all other (super)fields of the 2DTL (STL) hierarchy. Keeping in mind this correspondence it is quite natural to suppose that the algorithm [1, 2, 3] of constructing symmetries of the 2DTL and $N = (1|1)$ 2DTL equations is encoded in the structure of these hierarchies. In the present paper, we demonstrate by explicit construction that this is indeed the case at least with respect to bosonic symmetries.

The paper is organized as follows. In sections 2 and 3, we reproduce the algorithm [1, 2, 3] of constructing bosonic symmetries of the 2DTL and $N = (1|1)$ 2DTL equations starting with the 2DTL [14] and STL [15] hierarchies, respectively, following the methodology of operating with difference equations developed in [5, 6, 7, 8, 9, 10]. Furthermore, we also establish algebras of both fermionic and bosonic symmetries which were only conjectured in [1, 2, 3], discuss peculiarities of constructing fermionic symmetries as well as propose related new problems to be solved in future. In section 3, as a byproduct we also prove the proposed in [16] conjecture

regarding an $N = (2|2)$ superfield formulation of the STL hierarchy. In section 4, we solve the problem of constructing solutions to the symmetry equation corresponding to the $N = (0|2)$ supersymmetric 2DTL. Thus, we first propose the new $N = (0|2)$ supersymmetric 2DTL hierarchy which contains the $N = (0|2)$ 2DTL equation and then construct both bosonic and fermionic symmetries of the latter equation as well as their algebra. We also discuss an $N = (0|2)$ superfield formulation of the $N = (0|2)$ 2DTL hierarchy. Section 5 is devoted to a generalization: We propose a wide class of new supersymmetric integrable hierarchies whose first representative is the $N = (0|2)$ 2DTL hierarchy. In section 6, we present a short summary of the main results obtained in the paper.

2 2DTL hierarchy and symmetries of 2DTL equation

In this section, based on the 2DTL hierarchy of ref. [14], we develop a general scheme for constructing symmetries of the 2DTL equation and their algebra and as a byproduct establish the origin of the symmetries constructed in [1].

2.1 Lax pair representation and flows

Our starting point is the Lax pair representation of the 2DTL hierarchy [14, 17]:

$$\begin{aligned}\partial_n^\pm L^+ &= [((L^\pm)^n)_\pm, L^+], \\ \partial_n^\pm L^- &= [((L^\pm)^n)_\pm, L^-], \quad n \in \mathbb{N},\end{aligned}\tag{1}$$

with the two Lax operators L^+ and L^- ,

$$L^+ = \sum_{k=0}^{\infty} u_{k,j} e^{(1-k)\partial}, \quad L^- = \sum_{k=0}^{\infty} v_{k,j} e^{(k-1)\partial},\tag{2}$$

$$u_{0,j} \equiv 1, \quad v_{0,j} \neq 0,\tag{3}$$

which generates the abelian algebra of the flows

$$[\partial_n^\pm, \partial_l^\pm] = [\partial_n^+, \partial_l^-] = 0. \quad (4)$$

Here, the bosonic fields $u_{k,j} \equiv u_{k,j}(\{t_n^+, t_n^-\})$ and $v_{k,j} \equiv v_{k,j}(\{t_n^+, t_n^-\})$ are defined on the lattice, $j \in \mathbb{Z}$, and t_n^\pm are evolution times; $\partial_n^\pm := \frac{\partial}{\partial t_n^\pm}$ and the subscript $+$ ($-$) means the part of an operator which includes operators $e^{l\partial}$ at $l \geq 0$ ($l < 0$). Hereafter, the operator $e^{l\partial}$ ($l \in \mathbb{Z}$) is the discrete lattice shift which acts according to the rule

$$e^{l\partial} u_{k,j} \equiv u_{k,j+l} e^{l\partial}, \quad e^{l\partial} v_{k,j} \equiv v_{k,j+l} e^{l\partial}. \quad (5)$$

In this section, we will also use the following useful notation

$$(L^+)^m := \sum_{k=0}^{\infty} u_{k,j}^{(m)} e^{(m-k)\partial}, \quad (L^-)^m := \sum_{k=0}^{\infty} v_{k,j}^{(m)} e^{(k-m)\partial}, \quad (6)$$

where $\{u_{k,j}^{(m)}, v_{k,j}^{(m)}\}$ ($u_{k,j}^{(1)} \equiv u_{k,j}$, $v_{k,j}^{(1)} \equiv v_{k,j}$) are the functionals of the original fields $\{u_{k,j}, v_{k,j}\}$ whose explicit form is not important for further consideration but the explicit form of the following functionals:

$$u_{0,j}^{(m)} = 1 \quad (7)$$

which can easily be found using eqs. (3).

The following set of operator equations:

$$\begin{aligned} \partial_n^\pm (L^+)^m &= [((L^\pm)^n)_\pm, (L^+)^m], \\ \partial_n^\pm (L^-)^m &= [((L^\pm)^n)_\pm, (L^-)^m], \quad n, m \in \mathbb{N} \end{aligned} \quad (8)$$

is identically satisfied as a consequence of eqs. (1), and the corresponding system of evolution equations for the functionals $\{u_{k,j}^{(m)}, v_{k,j}^{(m)}\}$ can easily be derived from them. It reads

$$\partial_n^+ u_{k,j}^{(m)} = \sum_{p=0}^n (u_{p,j}^{(n)} u_{k-p+n,j-p+n}^{(m)} - u_{p,j-k+p-n+m}^{(n)} u_{k-p+n,j}^{(m)}), \quad (9)$$

$$\partial_n^- u_{k,j}^{(m)} = \sum_{p=0}^{n-1} (v_{p,j}^{(n)} u_{k+p-n,j+p-n}^{(m)} - v_{p,j-k-p+n+m}^{(n)} u_{k+p-n,j}^{(m)}), \quad (10)$$

$$\partial_n^+ v_{k,j}^{(m)} = \sum_{p=0}^n (u_{p,j}^{(n)} v_{k+p-n,j-p+n}^{(m)} - u_{p,j+k+p-n-m}^{(n)} v_{k+p-n,j}^{(m)}), \quad (11)$$

$$\partial_n^- v_{k,j}^{(m)} = \sum_{p=0}^{n-1} (v_{p,j}^{(n)} v_{k-p+n,j+p-n}^{(m)} - v_{p,j+k-p+n-m}^{(n)} v_{k-p+n,j}^{(m)}), \quad (12)$$

where all fields $\{u_{k,j}^{(m)}, v_{k,j}^{(m)}\}$ in the right-hand side should be put equal to zero at $k < 0$.

2.2 Symmetries of 2DTL equation

The 2DTL hierarchy (1–2) is a system of infinitely many equations for *infinitely many* fields $\{u_{k,j}, v_{k,j}\}$, while the 2DTL equation

$$\partial_1^- \partial_1^+ \ln v_{0,j} = -v_{0,j+1} + 2v_{0,j} - v_{0,j-1} \quad (13)$$

represents its first flow and involves only the *single* lattice field $v_{0,j}$ and two evolution derivatives, ∂_1^- and ∂_1^+ . The 2DTL equation (13) can be read from eqs. (10–11) if eqs. (10) are restricted to the values $\{n = m = k = 1\}$

$$\partial_1^- u_{1,j} = v_{0,j} - v_{0,j+1} \quad (14)$$

and eqs. (11) to $\{n = m = 1, k = 0\}$

$$\partial_1^+ v_{0,j} = v_{0,j}(u_{1,j} - u_{1,j-1}), \quad (15)$$

and then the field $u_{1,j}$ is eliminated from eqs. (14–15).

Now we would like to demonstrate that symmetries of the 2DTL equation (13) can be decoded from the system (9–12).

First, one can easily observe the existence of the following subalgebra of the flow algebra (4):

$$[\partial_1^+, \partial_n^\pm] = [\partial_1^-, \partial_n^\pm] = 0 \quad (16)$$

which is valid by construction of the 2DTL hierarchy, i.e. the flows ∂_n^\pm of the 2DTL hierarchy commute simultaneously with both the derivatives (∂_1^+ and ∂_1^-) entering into the 2DTL equation (13). The latter remarkable

property is a necessary, but not sufficient condition for the flows ∂_n^\pm to form symmetries of the 2DTL equation (13). Nevertheless, keeping in mind this property it is quite natural to suppose that the flows ∂_n^\pm form the symmetries (see, ref. [17] for slightly different arguments), although to complete the proof, we have additionally to show that they can in fact be realized in terms of the 2DTL field $v_{0,j}$ alone.

Second, with the last goal in mind let us discuss the above-mentioned candidates to be the symmetries,

$$\partial_n^+ v_{0,j} = +v_{0,j}(u_{n,j}^{(n)} - u_{n,j-1}^{(n)}) \quad (17)$$

and

$$\partial_n^- v_{0,j} = -v_{0,j}(v_{n,j}^{(n)} - v_{n,j-1}^{(n)}), \quad (18)$$

in more detail. When deriving these equations, we have used eqs. (11–12) at $\{m = 1, k = 0\}$ and the useful relation

$$\sum_{p=0}^n (v_{p,j}^{(n)} v_{n-p,j+p-n} - v_{p,j+n-p-1}^{(n)} v_{n-p,j}) = 0 \quad (19)$$

resulting from the simple, obvious identity $(L^-)^n L^- - L^-(L^-)^n = 0$. It turns out that the functionals $u_{n,j}^{(n)}$ and $v_{n,j}^{(n)}$ in the right-hand side of eqs. (17) and (18) can in fact be expressed in terms of the field $v_{0,j}$ alone by excluding all other fields of the 2DTL hierarchy by means of the flows ∂_1^- and ∂_1^+ , respectively, entering into the system (9–12). In order to see that, let us analyze eqs. (10) and (11), respectively, at $n = 1$

$$\partial_1^- u_{k,j}^{(m)} = v_{0,j} u_{k-1,j-1}^{(m)} - v_{0,j-k+m+1} u_{k-1,j}^{(m)}, \quad (20)$$

$$\partial_1^+ v_{k,j}^{(m)} - v_{k,j}^{(m)} (u_{1,j} - u_{1,j+k-m}) = v_{k-1,j+1}^{(m)} - v_{k-1,j}^{(m)}. \quad (21)$$

Equation (21) can easily be transformed into a more useful form for a further analysis which is similar to eq. (20). Thus, using the relations

$$u_{1,j} - u_{1,j+k-m} = +\partial_1^+ \ln \prod_{n=1}^{m-k} v_{0,j+k-m+n}, \quad m > k \quad (22)$$

resulting from eq. (15) and introducing the new basis $v_{k,j}^{(m)} \Rightarrow \tilde{v}_{k,j}^{(m)}$, according to the formulae

$$v_{m,j}^{(m)} = \tilde{v}_{m,j}^{(m)}, \quad v_{k,j}^{(m)} = \tilde{v}_{k,j}^{(m)} \prod_{n=1}^{m-k} v_{0,j+k-m+n} \quad m > k, \quad (23)$$

eq. (21) becomes

$$\partial_1^+ \tilde{v}_{k,j}^{(m)} = v_{0,j+1} \tilde{v}_{k-1,j+1}^{(m)} - v_{0,j+k-m} \tilde{v}_{k-1,j}^{(m)}, \quad m \geq k. \quad (24)$$

A simple inspection of (20) and (24) shows that they in fact allow one to express $u_{n,j}^{(n)}$ and $v_{n,j}^{(n)}$ in terms of $v_{0,j}$. Indeed, eq. (20) ((24)) represents a recurrent relation connecting the functional $u_{k,j}^{(n)}$ ($\tilde{v}_{k,j}^{(n)}$) with $u_{k-1,i}^{(n)}$ ($\tilde{v}_{k-1,i}^{(n)}$). Being iterated with the simple starting value $u_{0,j}^{(n)} = 1$ (7) ($\tilde{v}_{0,j}^{(n)} = 1$), it generates a very nontrivial expression for the functional $u_{n,j}^{(n)}$ ($v_{n,j}^{(n)} \equiv \tilde{v}_{n,j}^{(n)}$) in terms of $v_{0,j}$ after the n -th step of the iteration procedure. The latter, in turn, yield the symmetries $\partial_n^+ v_{0,j}$ and $\partial_n^- v_{0,j}$ to the 2DTL equation (13) via eqs. (17) and (18).

Let us remark that the 2DTL equation (13) possesses the involution

$$(\partial_1^\pm)^* = \partial_1^\mp, \quad (v_{0,j})^* = v_{0,i-j}, \quad (25)$$

which relates the symmetries (17), (20) with the symmetries (18), (23–24), according to the following rule:

$$(\partial_n^\pm)^* = \partial_n^\mp, \quad (u_{k,j}^{(m)})^* = \tilde{v}_{k,i-j-1}^{(m)}, \quad (\tilde{v}_{k,j}^{(m)})^* = u_{k,i-j-1}^{(m)}, \quad (26)$$

where $i \in \mathbb{Z}$ is a fixed number. Besides the involution (25–26), there exists also another involution

$$(\partial_n^\pm)^\bullet = \partial_n^\pm, \quad (v_{0,j})^\bullet = v_{0,i-j}. \quad (27)$$

Applying this involution to equations (18) and (24) and introducing the following notation:

$$(\tilde{v}_{k,j}^{(m)})^\bullet := u_{k,i-j-1}^{(m)-}, \quad u_{k,j}^{(m)} := u_{k,j}^{(m)+}, \quad (28)$$

equations (17–18), (20) and (24) can be rewritten in the following unified form:

$$\begin{aligned} \partial_n^\pm v_{0,j} &= v_{0,j} (u_{n,j}^{(n)\pm} - u_{n,j-1}^{(n)\pm}), \\ \partial_1^\mp u_{k,j}^{(n)\pm} &= v_{0,j} u_{k-1,j-1}^{(n)\pm} - v_{0,j-k+n+1} u_{k-1,j}^{(n)\pm}, \quad u_{0,j}^{(n)\pm} = 1. \end{aligned} \quad (29)$$

The symmetries (29) of the 2DTL equation (13) reproduce the solutions to the corresponding symmetry equation derived first in [1] by a rather heuristic construction, while the algebra (4) of the symmetries was not established there. One of the advantages of the general, algorithmic procedure, developed here, is the derivation of both the symmetries (29) and their algebra (4). As a byproduct we have also established the origin of the symmetries discussed in [1].

In next sections we extend the above-developed scheme of constructing symmetries of the 2DTL equation to the case of supersymmetric Toda lattices and discuss supersymmetric peculiarities related to fermionic flows and their algebras.

3 $N=(2|2)$ supersymmetric 2DTL hierarchy

In this section, we establish the relationship between bosonic symmetries of the $N = (1|1)$ supersymmetric 2DTL equation constructed in [2] and bosonic flows of the STL hierarchy of ref. [15]. We also discuss the subtle point for a relation between fermionic symmetries of the $N = (1|1)$ 2DTL equation constructed in [3] and fermionic flows of the STL hierarchy [15], and establish the algebra of both bosonic and fermionic symmetries.

3.1 Lax pair representation and flows

The Lax pair representation of the STL hierarchy is [15]

$$\begin{aligned} D_n^\pm L^+ &= (-1)^n (((L^\pm)_*)_\pm)^* L^+ - (L^+)^{*(n)} ((L^\pm)_*)_\pm \\ &\quad + \frac{1}{2}(1 \pm 1)(1 - (-1)^n)(L^+)_*^{n+1}, \\ D_n^\pm L^- &= (-1)^n (((L^\pm)_*)_\pm)^* L^- - (L^-)^{*(n)} ((L^\pm)_*)_\pm \\ &\quad + \frac{1}{2}(1 \mp 1)(1 - (-1)^n)(L^-)_*^{n+1}, \quad n \in \mathbb{N}, \end{aligned} \quad (30)$$

$$L^+ = \sum_{k=0}^{\infty} u_{k,j} e^{(1-k)\partial}, \quad L^- = \sum_{k=0}^{\infty} v_{k,j} e^{(k-1)\partial}, \quad (31)$$

$$u_{0,j} \equiv 1, \quad v_{0,j} \neq 0, \quad (32)$$

and it generates the non-abelian algebra of the flows

$$[D_n^+, D_l^-] = [D_n^\pm, D_{2l}^\pm] = 0, \quad \{D_{2n+1}^\pm, D_{2l+1}^\pm\} = 2D_{2(n+l+1)}^\pm \quad (33)$$

which may be realized via

$$D_{2n}^\pm = \partial_{2n}^\pm, \quad D_{2n+1}^\pm = \partial_{2n+1}^\pm + \sum_{l=1}^{\infty} t_{2l-1}^\pm \partial_{2(k+l)}^\pm, \quad (34)$$

where D_{2n}^\pm and t_{2n}^\pm (D_{2n+1}^\pm and t_{2n+1}^\pm) are bosonic (fermionic) evolution derivatives and times, respectively; $u_{2k,j}(\{t_n^+, t_n^-\})$ and $v_{2k,j}(\{t_n^+, t_n^-\})$ ($u_{2k+1,j}(\{t_n^+, t_n^-\})$ and $v_{2k+1,j}(\{t_n^+, t_n^-\})$) are bosonic (fermionic) lattice fields. Hereafter, the subscripts (superscripts) $*$ ($*(n)$ and $*$) are defined according to the rule [15]:

$$\begin{aligned} (L^\pm)_*^{2n} &:= ((L^\pm)^* L^\pm)^n, & (L^\pm)_*^{2n+1} &:= L^\pm ((L^\pm)^* L^\pm)^n, \\ (L^\pm)^{*(2n)} &:= L^\pm, & (L^\pm)^{*(2n+1)} &:= (L^\pm)^*, \\ (L^\pm[u_{k,j}, v_{k,j}])^* &:= L^\pm[u_{k,j}^*, v_{k,j}^*], & (u_{k,j}, v_{k,j})^* &:= (-1)^k (u_{k,j}, v_{k,j}). \end{aligned} \quad (35)$$

In this section we will use the following notation:

$$(L^+)_*^m := \sum_{k=0}^{\infty} u_{k,j}^{(m)} e^{(m-k)\partial}, \quad (L^-)_*^m := \sum_{k=0}^{\infty} v_{k,j}^{(m)} e^{(k-m)\partial}, \quad (36)$$

where $\{u_{2k,j}^{(m)}, v_{2k,j}^{(m)}\}$ and $\{u_{2k+1,j}^{(m)}, v_{2k+1,j}^{(m)}\}$ ($u_{k,j}^{(1)} \equiv u_{k,j}$, $v_{k,j}^{(1)} \equiv v_{k,j}$) are bosonic and fermionic functionals of the original fields $\{u_{k,j}, v_{k,j}\}$ whose explicit form is not important in what follows but

$$u_{0,j}^{(m)} = 1. \quad (37)$$

The operator equations

$$\begin{aligned} D_n^\pm (L^+)_*^m &= (-1)^{nm} (((L^\pm)_*^n)_\pm)^{*(m)} (L^+)_*^m - ((L^+)_*^m)^{*(n)} ((L^\pm)_*^n)_\pm \\ &\quad + \frac{1}{2} (1 \pm 1) (1 - (-1)^{nm}) (L^+)_*^{n+m}, \\ D_n^\pm (L^-)_*^m &= (-1)^{nm} (((L^\pm)_*^n)_\pm)^{*(m)} (L^-)_*^m - ((L^-)_*^m)^{*(n)} ((L^\pm)_*^n)_\pm \\ &\quad + \frac{1}{2} (1 \mp 1) (1 - (-1)^{nm}) (L^-)_*^{n+m}, \quad n, m \in \mathbb{N} \end{aligned} \quad (38)$$

are identically satisfied on the shell of the original equations (30) and reproduce the latter at the value $m = 1$. Let us remark that they can identically be rewritten in a rather standard Lax pair form if artificial fermionic parameters ϵ_n and ε_m are introduced,

$$\begin{aligned} \left(\prod_{k=1}^n \epsilon_k \right) D_n^\pm \prod_{p=1}^m (\varepsilon_p L^\mp) &= \left[\left(\prod_{k=1}^n (\epsilon_k L^\pm) \right)_\pm, \prod_{p=1}^m (\varepsilon_p L^\mp) \right], \\ \left(\prod_{k=1}^n \epsilon_k \right) D_n^\pm \prod_{p=1}^m (\varepsilon_p L^\pm) &= (-1)^{nm} \\ &\times \left[\left(\prod_{k=1}^n (\epsilon_k L^\pm) \right)_{\pm(-1)^{nm}}, \prod_{p=1}^m (\varepsilon_p L^\pm) \right]. \end{aligned} \quad (39)$$

The flows for the functionals $\{u_{k,j}^{(m)}, v_{k,j}^{(m)}\}$, corresponding to eqs. (38), are

$$\begin{aligned} D_n^+ u_{k,j}^{(2m)} &= \sum_{p=0}^n (u_{p,j}^{(n)} u_{k-p+n,j-p+n}^{(2m)} \\ &\quad - (-1)^{(p+n)(k-p+n)} u_{p,j-k+p-n+2m}^{(n)} u_{k-p+n,j}^{(2m)}), \end{aligned} \quad (40)$$

$$\begin{aligned} D_{2n}^+ u_{k,j}^{(2m+1)} &= \sum_{p=0}^{2n} ((-1)^p u_{p,j}^{(2n)} u_{k-p+2n,j-p+2n}^{(2m+1)} \\ &\quad - (-1)^{p(k-p)} u_{p,j-k+p-2n+2m+1}^{(2n)} u_{k-p+2n,j}^{(2m+1)}), \end{aligned} \quad (41)$$

$$\begin{aligned} D_{2n+1}^+ u_{k,j}^{(2m+1)} &= \sum_{p=1}^k ((-1)^{p+1} u_{p+2n+1,j}^{(2n+1)} u_{k-p,j-p}^{(2m+1)} \\ &\quad + (-1)^{p(k-p)} u_{p+2n+1,j-k+p+2m+1}^{(2n+1)} u_{k-p,j}^{(2m+1)}), \end{aligned} \quad (42)$$

$$\begin{aligned} D_n^- u_{k,j}^{(m)} &= \sum_{p=0}^{n-1} ((-1)^{(p+n)m} v_{p,j}^{(n)} u_{k+p-n,j+p-n}^{(m)} \\ &\quad - (-1)^{(p+n)(k+p-n)} v_{p,j-k-p+n+m}^{(n)} u_{k+p-n,j}^{(m)}), \end{aligned} \quad (43)$$

$$\begin{aligned} D_n^+ v_{k,j}^{(m)} &= \sum_{p=0}^n ((-1)^{(p+n)m} u_{p,j}^{(n)} v_{k+p-n,j-p+n}^{(m)} \\ &\quad - (-1)^{(p+n)(k+p-n)} u_{p,j+k+p-n-m}^{(n)} v_{k+p-n,j}^{(m)}), \end{aligned} \quad (44)$$

$$D_n^- v_{k,j}^{(2m)} = \sum_{p=0}^{n-1} (v_{p,j}^{(n)} v_{k-p+n,j+p-n}^{(2m)} - (-1)^{(p+n)(k-p+n)} v_{p,j+k-p+n-2m}^{(n)} v_{k-p+n,j}^{(2m)}), \quad (45)$$

$$D_{2n}^- v_{k,j}^{(2m+1)} = \sum_{p=0}^{2n-1} ((-1)^p v_{p,j}^{(2n)} v_{k-p+2n,j+p-2n}^{(2m+1)} - (-1)^{p(k-p)} v_{p,j+k-p+2n-2m-1}^{(2n)} v_{k-p+2n,j}^{(2m+1)}), \quad (46)$$

$$D_{2n+1}^- v_{k,j}^{(2m+1)} = \sum_{p=0}^k ((-1)^{p+1} v_{p+2n+1,j}^{(2n+1)} v_{k-p,j+p}^{(2m+1)} + (-1)^{p(k-p)} v_{p+2n+1,j+k-p-2m-1}^{(2n+1)} v_{k-p,j}^{(2m+1)}), \quad (47)$$

where all fields $\{u_{k,j}^{(m)}, v_{k,j}^{(m)}\}$ in the right-hand side should be put equal to zero at $k < 0$. When deriving eqs. (42) and (47) we have used the following identity: $2(L^\pm)_*^{2(n+m+1)} = ((L^\pm)_*^{2n+1})^* (L^\pm)_*^{2m+1} + ((L^\pm)_*^{2m+1})^* (L^\pm)_*^{2n+1}$ which can easily be verified using the definitions (35).

3.2 Bosonic symmetries of $N = (2|2)$ 2DTL equation

The $N = (2|2)$ supersymmetric 2DTL equation belongs to the system of equations (40–47). In order to see that, let us consider eqs. (43) at $\{n = m = k = 1\}$

$$D_1^- u_{1,j} = -v_{0,j} - v_{0,j+1} \quad (48)$$

and eqs. (44) at $\{n = m = 1, k = 0\}$

$$D_1^+ v_{0,j} = v_{0,j}(u_{1,j} - u_{1,j-1}). \quad (49)$$

Then, eliminating the field $u_{1,j}$ from eqs. (48–49) we obtain

$$D_1^+ D_1^- \ln v_{0,j} = v_{0,j+1} - v_{0,j-1}. \quad (50)$$

Equation (50) reproduces the $N = (1|1)$ superfield form of the $N = (2|2)$ superconformal 2DTL equation (see, e.g. refs. [18, 4] and references

therein) which is the supersymmetrization of the system of two decoupled 2DTL equations (13). Indeed, in terms of the superfield components

$$f_j \equiv v_{0,j}|, \quad \gamma_j^\pm \equiv (\mathcal{D}_1^\pm \ln v_{0,j})|, \quad (51)$$

where f_j ($\gamma_j, \bar{\gamma}_j$) are bosonic (fermionic) fields and $|$ means the $t_1^+ \rightarrow 0$ limit, eq. (50) becomes

$$\begin{aligned} \partial_1^+ \partial_1^- \ln f_j &= -f_{j+1}f_{j+2} + f_j(f_{j+1} + f_{j-1}) - f_{j-1}f_{j-2} \\ &\quad - f_{j+1}\gamma_{j+1}^+ \gamma_{j+1}^- + f_{j-1}\gamma_{j-1}^+ \gamma_{j-1}^-, \\ \mp \partial_\mp \gamma_j^\pm &= f_{j+1}\gamma_{j+1}^\mp - f_{j-1}\gamma_{j-1}^\mp. \end{aligned} \quad (52)$$

Then, denoting $v_j := f_{j-1}f_j$ in the bosonic limit when all fermionic fields are set to zero, we finally obtain the equation

$$\partial_1^- \partial_1^+ \ln v_j = -v_{j+2} + 2v_j - v_{j-2} \quad (53)$$

which obviously splits into the system of two decoupled 2DTL equations (13) for the functions at even and odd lattice points, i.e. v_{2j} and v_{2j+1} .

Now, we would like to discuss how bosonic symmetries of the $N = (2|2)$ 2DTL equation (50) originate from the system (40–47). It appears that the approach developed in the previous section for the case of symmetries of the bosonic 2DTL equation (13) can straightforwardly be extended to the present case.

First, let us derive the flows $D_n^+ v_{0,j}$ and $D_n^- v_{0,j}$ of the STL hierarchy considering eqs. (44) at $\{m = 1, k = 0\}$ and eqs. (46–47) at $\{m = k = 0\}$,

$$D_n^+ v_{0,j} = +v_{0,j}(u_{n,j}^{(n)} - u_{n,j-1}^{(n)}) \quad (54)$$

and

$$D_n^- v_{0,j} = -v_{0,j}(v_{n,j}^{(n)} - v_{n,j-1}^{(n)}), \quad (55)$$

respectively. When calculating eqs. (55), we have used the relation

$$\sum_{p=0}^{2n} ((-1)^p v_{p,j}^{(2n)} v_{2n-p,j+p-2n} - (-1)^{p^2} v_{p,j+2n-p-1}^{(2n)} v_{2n-p,j}) = 0 \quad (56)$$

resulting from the identity $((L^-)_*^{2n})^* L^- - L^- (L^-)_*^{2n} = 0$ following from the definitions (35).

From the algebra (33) one can easily observe that only bosonic flows D_{2n}^\pm commute simultaneously with both the fermionic derivatives D_1^+ and D_1^- entering into the $N = (2|2)$ 2DTL equation (50),

$$[D_1^+, D_{2n}^\pm] = [D_1^-, D_{2n}^\pm] = 0, \quad (57)$$

while the fermionic flows D_{2n+1}^\pm do not satisfy this property. Due to this reason, the bosonic flows D_{2n}^\pm (54–55) form symmetries of the $N = (1|1)$ 2DTL equation (50) if one can possibly express the functionals $u_{2n,j}^{(2n)}$ and $v_{2n,j}^{(2n)}$ in the right-hand side of eqs. (54) and (55) in terms of the field $v_{0,j}$ alone, while the fermionic flows D_{2n+1}^\pm do not.

With the aim to express $u_{n,j}^{(n)}$ and $v_{n,j}^{(n)}$ in terms of $v_{0,j}$, let us consider eqs. (43) and (44) at $n = 1$

$$D_1^- u_{k,j}^{(m)} = (-1)^m v_{0,j} u_{k-1,j-1}^{(m)} + (-1)^k v_{0,j-k+m+1} u_{k-1,j}^{(m)}, \quad (58)$$

$$D_1^+ v_{k,j}^{(m)} - (-1)^k v_{k,j}^{(m)} (u_{1,j} - u_{1,j+k-m}) = (-1)^m v_{k-1,j+1}^{(m)} + (-1)^k v_{k-1,j}^{(m)}. \quad (59)$$

Substituting

$$u_{1,j} - u_{1,j+k-m} = +D_1^+ \ln \prod_{n=1}^{m-k} v_{0,j+k-m+n}, \quad m > k \quad (60)$$

derived from eq. (49) into eq. (59) and introducing the new basis $v_{k,j}^{(m)} \Rightarrow \tilde{v}_{k,j}^{(m)}$, according to the formulae

$$v_{m,j}^{(m)} = \tilde{v}_{m,j}^{(m)}, \quad v_{k,j}^{(m)} = \tilde{v}_{k,j}^{(m)} \prod_{n=1}^{m-k} v_{0,j+k-m+n} \quad m > k, \quad (61)$$

eq. (59) becomes

$$D_1^+ \tilde{v}_{k,j}^{(m)} = (-1)^m v_{0,j+1} \tilde{v}_{k-1,j+1}^{(m)} + (-1)^k v_{0,j+k-m} \tilde{v}_{k-1,j}^{(m)}, \quad m \geq k \quad (62)$$

and has the form similar to eq. (58).

The equations (58) and (62) derived represent recurrent relations which being iterated with the starting values $u_{0,j}^{(n)} = 1$ (37) and $\tilde{v}_{0,j}^{(n)} = 1$ allow one to express the functionals $u_{n,j}^{(n)}$ and $v_{n,j}^{(n)} \equiv \tilde{v}_{n,j}^{(n)}$ in terms of $v_{0,j}$ after the n -th

step of the iteration procedure. The latter yield the bosonic symmetries $D_{2n}^+ v_{0,j}$ and $D_{2n}^- v_{0,j}$ to the $N = (1|1)$ 2DTL equation (50) via eqs. (54) and (55).

Let us remark that the $N = (2|2)$ 2DTL equation (50) possesses the following involution:

$$(D_1^\pm)^* = D_1^\mp, \quad v_{0,j}^* = v_{0,i-j} \quad (63)$$

which relates the flows (54), (58) with the flows (55), (61–62),

$$(D_n^\pm)^* = D_n^\mp, \quad (u_{k,j}^{(m)})^* = \tilde{v}_{k,i-j-1}^{(m)}, \quad (\tilde{v}_{k,j}^{(m)})^* = u_{k,i-j-1}^{(m)}, \quad (64)$$

where $i \in \mathbb{Z}$ is a fixed number. Besides the involution (63–64), there exists also another involution

$$(D_n^\pm)^\bullet = D_n^\pm, \quad (v_{0,j})^\bullet = -v_{0,i-j}. \quad (65)$$

Applying the latter to (55) and (62) and introducing the notation

$$(\tilde{v}_{k,j}^{(m)})^\bullet := u_{k,i-j-1}^{(m)-}, \quad u_{k,j}^{(m)} := u_{k,j}^{(m)+}, \quad (66)$$

the flows (54–55), (58) and (62) finally become

$$\begin{aligned} D_n^\pm v_{0,j} &= v_{0,j} (u_{n,j}^{(n)\pm} - u_{n,j-1}^{(n)\pm}), \quad u_{0,j}^{(n)\pm} = 1, \\ \pm D_1^\mp u_{k,j}^{(n)\pm} &= (-1)^n v_{0,j} u_{k-1,j-1}^{(n)\pm} - (-1)^k v_{0,j-k+n+1} u_{k-1,j}^{(n)\pm}. \end{aligned} \quad (67)$$

The bosonic flows D_{2n}^\pm , resulting from eqs. (67),

$$\begin{aligned} D_{2n}^\pm v_{0,j} &= v_{0,j} (u_{2n,j}^{(2n)\pm} - u_{2n,j-1}^{(2n)\pm}), \quad u_{0,j}^{(2n)\pm} = 1, \\ \pm D_1^\mp u_{k,j}^{(2n)\pm} &= v_{0,j} u_{k-1,j-1}^{(2n)\pm} - (-1)^k v_{0,j-k+2n+1} u_{k-1,j}^{(2n)\pm}, \end{aligned} \quad (68)$$

reproduce the bosonic solutions to the symmetry equation corresponding to the $N = (1|1)$ 2DTL equation (50) derived in [2] by a rather heuristic construction, while the algebra of the bosonic symmetries D_{2n}^\pm (68)

$$[D_{2n}^\pm, D_{2l}^\pm] = [D_{2n}^+, D_{2l}^-] = 0 \quad (69)$$

resulting from eqs. (33) was not proved there.

3.3 Fermionic symmetries of $N = (2|2)$ 2DTL equation

In this subsection, we discuss the origin of the fermionic symmetries, proposed in [3], of the $N = (1|1)$ 2DTL equation (50) and construct their algebra.

For completeness, we would like to start with the derivation of a close set of equations for the functionals $u_{k,j}^{(2n)}$ aiming to reproduce the solutions corresponding to fermionic symmetries first observed in [3].

With this goal in mind, let us consider eqs. (40) at $n = 1$,

$$D_1^+ u_{k,j}^{(2n)} + (-1)^k u_{k,j}^{(2n)} (u_{1,j-k+2n} - u_{1,j}) = u_{k+1,j+1}^{(2n)} + (-1)^k u_{k+1,j}^{(2n)}. \quad (70)$$

Then, using the recursive substitution (58), we express the functionals $u_{k+1,j}^{(2n)}$ in the right-hand side of eqs. (70) in terms of the functionals $u_{k,j}^{(2n)}$; particularly, we also use the relation

$$u_{1,j-k+2n} - u_{1,j} = (D_1^-)^{-1} (v_{0,j} + v_{0,j+1} - v_{0,j-k+2n} - v_{0,j-k+2n+1}), \quad (71)$$

and as a result, we elaborate the following close equations for the functionals $u_{k,j}^{(2n)}$ at different lattice points $(j-1, j$ and $j+1)$, but with the same subscript k

$$\begin{aligned} & (-1)^k D_1^+ u_{k,j}^{(2n)} + u_{k,j}^{(2n)} (D_1^-)^{-1} (v_{0,j} - v_{0,j-k+2n+1} \\ & \quad + v_{0,j+1} - v_{0,j-k+2n}) \\ & = (D_1^-)^{-1} (v_{0,j} u_{k,j-1}^{(2n)} - v_{0,j-k+2n+1} u_{k,j+1}^{(2n)} \\ & \quad + (-1)^k (v_{0,j+1} - v_{0,j-k+2n}) u_{k,j}^{(2n)}) \end{aligned} \quad (72)$$

which reproduce the corresponding equations derived by a heuristic construction in [2, 3]. According to [2, 3], equations (72) can be treated as the result of the application of the recursive chain of substitutions (58) to the symmetry equation corresponding to the symmetries D_{2n}^+ (68) of the $N = (2|2)$ 2DTL equation (50). In other words, equations (72) represent the consistency conditions for the algebra (57) realized on the shell of the $N = (2|2)$ 2DTL equation (50). Due to this reason, we can forget for a moment about their hierarchy origin and discuss their solutions which will be relevant for further consideration.

At $k = 0$, equation (72) possesses a very simple, constant solution $u_{0,j}^{(2n)} = 1$ [2] which reproduces the condition (37) for the hierarchy we started with. As it has already been explained in the previous subsection, this solution generates a very non-trivial solution for the functional $u_{2n,j}^{(2n)}$ via eqs. (58) as well as the bosonic symmetry $D_{2n}^+ v_{0,j}$ to the $N = (1|1)$ 2DTL equation (50) via eq. (54).

It turns out that eq. (72) possesses also a fermionic, lattice-dependent solution at $k = -1$, namely [3]

$$u_{-1,j}^{(2n)} = (-1)^{j+1} \epsilon, \quad (73)$$

where ϵ is a dimensionless fermionic constant. It remains to show how fermionic symmetries are being activated. With this goal in mind, let us represent the bosonic time derivative D_{2n}^+ corresponding to the solution (73) and the functional $u_{k,j}^{(2n)}$ which enter eqs. (54), (58), (73) and (57) in the following form:

$$D_{2n}^+ := \epsilon \mathcal{D}_{2n+1}^+, \quad u_{k,j}^{(2n)} := \epsilon \mathcal{U}_{k+1,j}^{(2n+1)+} \quad (74)$$

defining a new fermionic evolution derivative \mathcal{D}_{2n+1}^+ and the functionals $\mathcal{U}_{k,j}^{(2n+1)+}$. Then, the fermionic constant ϵ enters linearly into both the sides of eqs. (54), (58), (73) and (57) which now become

$$\begin{aligned} \mathcal{D}_{2n+1}^\pm v_{0,j} &= v_{0,j} (\mathcal{U}_{2n+1,j}^{(2n+1)\pm} - \mathcal{U}_{2n+1,j-1}^{(2n+1)\pm}), \quad \mathcal{U}_{0,j}^{(2n+1)\pm} = (-1)^{j+1}, \\ \pm D_1^\mp \mathcal{U}_{k,j}^{(2n+1)\pm} &= -v_{0,j} \mathcal{U}_{k-1,j-1}^{(2n+1)\pm} + (-1)^k v_{0,j-k+2n+2} \mathcal{U}_{k-1,j}^{(2n+1)\pm}, \end{aligned} \quad (75)$$

$$\{D_1^+, \mathcal{D}_{2n+1}^\pm\} = \{D_1^-, \mathcal{D}_{2n+1}^\pm\} = 0. \quad (76)$$

When deriving eqs. (75–76) we have substituted eqs. (74) into eqs. (54), (58), (73) and (57), and additionally used the involution (65) and notation (66). Therefore, the flows \mathcal{D}_{2m+1}^\pm do not actually depend on ϵ , so ϵ is an artificial parameter which need not be introduced at all. The most important fact however is that \mathcal{D}_{2m+1}^\pm anticommute with the fermionic derivatives D_1^\pm (76) entering into the $N = (2|2)$ 2DTL equation (50) by construction, and due to this reason, they form symmetries of the $N = (2|2)$ 2DTL equation (50).

Although the existence of the symmetries \mathcal{D}_{2n+1}^\pm (75) was established in [3], their algebra was only conjectured by extending the algebra of a

few first bosonic and fermionic flows explicitly derived there. Now, we are ready to rigorously establish the algebra of all the bosonic \mathcal{D}_{2n}^\pm (68) and fermionic \mathcal{D}_{2n+1}^\pm (75) symmetries in the framework of the developed here approach.

Our strategy comprises a few steps.

First, let us calculate the fermionic symmetry $\mathcal{D}_1^\pm v_{0,j}$ (75) and its algebra expressing the symmetry in terms of the fermionic flow $D_1^\pm v_{0,j}$ (67) and using the algebra (33). They are

$$\mathcal{D}_1^\pm v_{0,j} \equiv (-1)^{j+1} D_1^\pm v_{0,j} \quad (77)$$

and

$$\{\mathcal{D}_1^\pm, \mathcal{D}_1^\pm\} v_{0,j} = -\{D_1^\pm, D_1^\pm\} v_{0,j} \equiv -2D_2^\pm v_{0,j}, \quad (78)$$

respectively.

Second, we use the derived relation (77) in order to replace D_1^\pm by \mathcal{D}_1^\pm in the expressions both for the bosonic (68) and fermionic (75) symmetries, then transform them to the new basis

$$\begin{aligned} \hat{u}_{k,j}^{(2n+1)\pm} &:= c_k (-1)^{(k+1)(j+1)} \mathcal{U}_{k,j}^{(2n+1)\pm}, & \hat{u}_{k,j}^{(2n)\pm} &:= c_k (-1)^{kj} u_{k,j}^{(2n)\pm}, \\ \hat{\mathcal{D}}_{2n+1}^\pm &:= c_{2n+1} \mathcal{D}_{2n+1}^\pm, & \hat{\mathcal{D}}_{2n}^\pm &:= c_{2n} D_{2n}^\pm, & c_{2n} = c_{2n+1} &\equiv (-1)^n \end{aligned} \quad (79)$$

which is defined by a single requirement that the form of the symmetries in this basis is as close as possible to the form of the flows D_n^\pm (67) of the STL hierarchy whose algebra (33) is known. In the new basis (79), the symmetries (68) and (75) as well the algebra (78) become

$$\begin{aligned} \hat{\mathcal{D}}_n^\pm v_{0,j} &= v_{0,j} (\hat{u}_{n,j}^{(n)\pm} - \hat{u}_{n,j-1}^{(n)\pm}), & \hat{u}_{0,j}^{(n)\pm} &= 1, \\ \pm \hat{\mathcal{D}}_1^\mp \hat{u}_{k,j}^{(n)\pm} &= (-1)^n v_{0,j} \hat{u}_{k-1,j-1}^{(n)\pm} + (-1)^k v_{0,j-k+n+1} \hat{u}_{k-1,j}^{(n)\pm} \end{aligned} \quad (80)$$

and

$$\{\hat{\mathcal{D}}_1^\mp, \hat{\mathcal{D}}_1^\mp\} = 2\hat{\mathcal{D}}_2^\mp, \quad (81)$$

respectively. When deriving the second line of eqs. (80), we have first acted by the fermionic derivative D_1^\mp on both sides of the second line of eqs. (68) and (75) and then used the latter once more together with eqs. (77–79) and

(81). A simple comparison allows one to immediately observe that (80) and (81) coincide with the expressions for the flows D_n^\pm (67) and the algebra of the derivatives D_1^\mp (78), respectively, where, however, the evolution derivatives D_n^\pm are replaced by \hat{D}_n^\pm . The obvious, important consequence from this observation is that the algebra of the evolution derivatives \hat{D}_n^\pm has also to reproduce the algebra of the evolution derivatives D_n^\pm (33). Thus, we are led to the following formulae for both this algebra and the algebras (78), (57) as well as (76) transformed to the basis (79)

$$[\hat{D}_n^+, \hat{D}_l^-] = [\hat{D}_n^\pm, \hat{D}_{2l}^\pm] = 0, \quad \{\hat{D}_{2n+1}^\pm, \hat{D}_{2l+1}^\pm\} = 2\hat{D}_{2(n+l+1)}^\pm \quad (82)$$

and

$$\begin{aligned} \{D_1^+, D_1^-\} &= 0, \quad \{D_1^\pm, D_1^\pm\} = -2\hat{D}_2^\pm, \\ [D_1^+, \hat{D}_n^\pm] &= [D_1^-, \hat{D}_n^\pm] = 0. \end{aligned} \quad (83)$$

By construction, the algebra (82) forms a symmetry algebra of the $N = (2|2)$ 2DTL equation (50). However, one can easily understand that the fermionic symmetries \mathcal{D}_{2n+1}^\pm (75) are also symmetries of the bosonic flows D_{2n}^\pm (68) of the STL hierarchy because of the following commutation relations:

$$[\mathcal{D}_{2n+1}^+, D_{2l}^\pm] = [\mathcal{D}_{2n+1}^-, D_{2l}^\pm] = 0 \quad (84)$$

resulting from the algebra (82) and the relations (79).

Let us also point out that bosonic and fermionic symmetries of the one-dimensional reduction of the $N = (2|2)$ 2DTL hierarchy — the $N = 4$ supersymmetric Toda chain hierarchy — were analyzed in detail in [19, 20].

The existence of the fermionic symmetries \mathcal{D}_{2n+1}^\pm (75) means that the Lax pair equations (30), we started with in this section, are not complete because they do not contain fermionic flows which would correspond to these symmetries. Therefore, the new problem arises: it would be interesting to construct both additional evolution equations for the Lax operators L^\pm (31) generated by the fermionic symmetries \mathcal{D}_{2n+1}^\pm (75) and commutation relations between the latter and the fermionic flows D_{2n+1}^\pm (67) of the STL hierarchy. The detailed analysis of this rather nontrivial problem is beyond the scope of the present paper and will be considered elsewhere.

Let us only mention that a similar task has partly been discussed in [21] in a slightly different context.

To close this section, let us briefly discuss one of the consequences of the results derived in this subsection which is important in the context of the problem of constructing an $N = (2|2)$ superfield formulation of the bosonic flows D_{2n}^\pm (68) of the STL hierarchy. Quite recently this problem was considered in [16] basing on the conjecture partly proved there (for more details, see ref. [16]). In terms of the objects introduced in the present paper this conjecture can be reformulated as a conjecture about the validity of the following constraints:

$$\left(\mathcal{D}_1^\mp + D_1^\mp\right) D_{2l}^\pm v_{0,2j} = 0, \quad \left(\mathcal{D}_1^\mp - D_1^\mp\right) D_{2l}^\pm v_{0,2j+1} = 0, \quad (85)$$

$$\left(\mathcal{D}_1^\pm + D_1^\pm\right) D_{2l}^\pm v_{0,2j} = 0, \quad \left(\mathcal{D}_1^\pm - D_1^\pm\right) D_{2l}^\pm v_{0,2j+1} = 0. \quad (86)$$

The proof that the constraints (85) are in fact satisfied is given in [16]. As concerns the remaining constraints (86), only evidence in their favour was presented there by confirming them (and (85)) explicitly for the first three bosonic flows D_{2n}^\pm ($n = 1, 2, 3$) from the set (68). Here, we are ready to prove this conjecture. Thus, using the relations (77) represented in the equivalent form

$$\left(\mathcal{D}_1^\mp + D_1^\mp\right) v_{0,2j} = 0, \quad \left(\mathcal{D}_1^\mp - D_1^\mp\right) v_{0,2j+1} = 0, \quad (87)$$

the constraints (85–86) can identically be rewritten in the following form more convenient for a further analysis:

$$[\mathcal{D}_1^\mp + D_1^\mp, D_{2l}^\pm] v_{0,2j} = 0, \quad [\mathcal{D}_1^\mp - D_1^\mp, D_{2l}^\pm] v_{0,2j+1} = 0, \quad (88)$$

$$[\mathcal{D}_1^\pm + D_1^\pm, D_{2l}^\pm] v_{0,2j} = 0, \quad [\mathcal{D}_1^\pm - D_1^\pm, D_{2l}^\pm] v_{0,2j+1} = 0. \quad (89)$$

It is a simple exercise now to verify that the correctness of the conjecture in the form of equations (88–89) is a direct consequence of the algebras (57) and (84).

4 N=(0|2) supersymmetric 2DTL hierarchy

In this section we propose the new, $N = (0|2)$ supersymmetric 2DTL hierarchy which includes the $N = (0|2)$ superconformal 2DTL equation derived in [4] and construct both bosonic and fermionic symmetries of the latter.

4.1 Lax pair representation and flows

Let us start with the following set of the consistent Lax pair equations:

$$\begin{aligned} D_n^+ L^+ &= (-1)^n (((L^+)_*^n)_+)^* L^+ - (L^+)^{*(n)} ((L^+)_*^n)_+ \\ &\quad + (1 - (-1)^n) (L^+)_*^{n+1}, \\ D_n^+ L^- &= ((L^+)_*^n)_+ L^- - (L^-)^{*(n)} ((L^+)_*^n)_+, \\ D_{2n}^- L^+ &= (((L^-)^n)_-)^* L^+ - (L^+) ((L^-)^n)_-, \\ D_{2n}^- L^- &= [((L^-)^n)_-, L^-], \quad n \in \mathbb{N}, \end{aligned} \quad (90)$$

$$L^+ = \sum_{k=0}^{\infty} u_{k,j} e^{(1-k)\partial}, \quad L^- = \sum_{k=0}^{\infty} v_{k,j} e^{(k-2)\partial}, \quad (91)$$

$$u_{0,j} \equiv 1, \quad v_{0,2j+1} \equiv 0, \quad v_{0,2j} \neq 0 \quad (92)$$

generating the non-abelian algebra of the flows

$$[D_n^+, D_{2l}^\pm] = [D_{2n}^-, D_{2l}^\pm] = 0, \quad \{D_{2n+1}^+, D_{2l+1}^+\} = 2D_{2(n+l+1)}^+ \quad (93)$$

which may be realized in the superspace $\{t_n^+, t_{2n}^-\}$

$$D_{2n}^\pm = \partial_{2n}^\pm, \quad D_{2n+1}^+ = \partial_{2n+1}^+ + \sum_{l=1}^{\infty} t_{2l-1}^+ \partial_{2(k+l)}^+, \quad (94)$$

where D_{2n}^\pm and t_{2n}^\pm (D_{2n+1}^+ and t_{2n+1}^+) are bosonic (fermionic) evolution derivatives and times, respectively; $u_{2k,j}(\{t_n^+, t_{2n}^-\})$ and $v_{2k,j}(\{t_n^+, t_{2n}^-\})$ ($u_{2k+1,j}(\{t_n^+, t_{2n}^-\})$ and $v_{2k+1,j}(\{t_n^+, t_{2n}^-\})$) are bosonic (fermionic) lattice fields.

In what follows we will show that the $N = (0|2)$ 2DTL equation [4] belongs to the set of equations (90) and due to this reason we call it the $N = (0|2)$ supersymmetric 2DTL hierarchy.

Let us introduce the following useful notation:

$$(L^+)_*^m := \sum_{k=0}^{\infty} u_{k,j}^{(m)} e^{(m-k)\partial}, \quad (L^-)^m := \sum_{k=0}^{\infty} v_{k,j}^{(m)} e^{(k-2m)\partial} \quad (95)$$

which will be used in this section. Here, $\{u_{2k,j}^{(m)}, v_{2k,j}^{(m)}\}$ and $\{u_{2k+1,j}^{(m)}, v_{2k+1,j}^{(m)}\}$ ($u_{k,j}^{(1)} \equiv u_{k,j}$, $v_{k,j}^{(1)} \equiv v_{k,j}$) are bosonic and fermionic functionals of the original fields $\{u_{k,j}, v_{k,j}\}$ whose explicit form is not important for the further consideration but the explicit form of the following functionals:

$$u_{0,j}^{(m)} = 1, \quad v_{0,2j+1}^{(m)} = 0, \quad v_{0,2j}^{(m)} \neq 0 \quad (96)$$

which can easily be found using eqs. (92).

One important remark is in order: the Lax pair representation (90–91) supplied by the constraints (92) cannot be obtained by reducting the Lax pair representation (30–32) of the $N = (2|2)$ 2DTL hierarchy. Indeed, if it would be the case, then the Lax operator L^- (91) had the square root of the form

$$(L^-) = ((L^-)^{\frac{1}{2}})^* (L^-)^{\frac{1}{2}}, \quad (L^-)^{\frac{1}{2}} = \sum_{k=0}^{\infty} v_{k,j}^{(\frac{1}{2})} e^{(k-1)\partial} \quad (97)$$

which reproduces the original Lax operator (31) of the $N = (2|2)$ 2DTL hierarchy, and as a consequence of eqs. (97), the field $v_{0,j}$ admits the following representation:

$$v_{0,j} = v_{0,j}^{(\frac{1}{2})} v_{0,j-1}^{(\frac{1}{2})}. \quad (98)$$

However, the latter is inconsistent with the conditions (92); so we come to the contradiction. Therefore, the conclusion is that the $N = (2|2)$ 2DTL hierarchy cannot be reduced to the $N = (0|2)$ 2DTL hierarchy.

The following operator equations:

$$\begin{aligned} D_n^+(L^+)_*^m &= (-1)^{nm} (((L^+)_*^n)_+)^*(m) (L^+)_*^m \\ &\quad - ((L^+)_*^m)^*(n) ((L^+)_*^n)_+ + (1 - (-1)^n) (L^+)_*^{n+m}, \\ D_n^+(L^-)^m &= ((L^+)_*^n)_+ (L^-)^m - ((L^-)^m)^*(n) ((L^+)_*^n)_+, \\ D_{2n}^-(L^+)_*^m &= (((L^-)^n)_-)^*(m) (L^+)_*^m - ((L^+)_*^m) ((L^-)^n)_-, \\ D_{2n}^-(L^-)^m &= [((L^-)^n)_-, (L^-)^m], \quad n, m \in \mathbb{N} \end{aligned} \quad (99)$$

are identically satisfied on the shell of the original equations (90), and the corresponding flows for the functionals $\{u_{k,j}^{(m)}, v_{k,j}^{(m)}\}$ are

$$D_n^+ u_{k,j}^{(2m)} = \sum_{p=0}^n (u_{p,j}^{(n)} u_{k-p+n,j-p+n}^{(2m)} - (-1)^{(p+n)(k-p+n)} u_{p,j-k+p-n+2m}^{(n)} u_{k-p+n,j}^{(2m)}), \quad (100)$$

$$D_{2n}^+ u_{k,j}^{(2m+1)} = \sum_{p=0}^{2n} ((-1)^p u_{p,j}^{(2n)} u_{k-p+2n,j-p+2n}^{(2m+1)} - (-1)^{p(k-p)} u_{p,j-k+p-2n+2m+1}^{(2n)} u_{k-p+2n,j}^{(2m+1)}), \quad (101)$$

$$D_{2n+1}^+ u_{k,j}^{(2m+1)} = \sum_{p=1}^k ((-1)^{p+1} u_{p+2n+1,j}^{(2n+1)} u_{k-p,j-p}^{(2m+1)} + (-1)^{p(k-p)} u_{p+2n+1,j-k+p+2m+1}^{(2n+1)} u_{k-p,j}^{(2m+1)}), \quad (102)$$

$$D_{2n}^- u_{k,j}^{(m)} = \sum_{p=0}^{2n-1} ((-1)^{pm} v_{p,j}^{(n)} u_{k+p-2n,j+p-2n}^{(m)} - (-1)^{p(k+p)} v_{p,j-k-p+2n+m}^{(n)} u_{k+p-2n,j}^{(m)}), \quad (103)$$

$$D_n^+ v_{k,j}^{(m)} = \sum_{p=0}^n (u_{p,j}^{(n)} v_{k+p-n,j-p+n}^{(m)} - (-1)^{(p+n)(k+p-n)} u_{p,j+k+p-n-2m}^{(n)} v_{k+p-n,j}^{(m)}), \quad (104)$$

$$D_{2n}^- v_{k,j}^{(m)} = \sum_{p=0}^{2n-1} (v_{p,j}^{(n)} v_{k-p+2n,j+p-2n}^{(m)} - (-1)^{p(k-p)} v_{p,j+k-p+2n-2m}^{(n)} v_{k-p+2n,j}^{(m)}), \quad (105)$$

where all fields $\{u_{k,j}^{(m)}, v_{k,j}^{(m)}\}$ in the right-hand side should be put equal to zero at $k < 0$.

4.2 Bosonic symmetries of $N = (0|2)$ 2DTL equation

Now, let us demonstrate how the $N = (0|2)$ 2DTL equation and its symmetries originate from this background.

With this goal in mind, let us consider eqs. (104) and (105) at $\{m = 1, k = 0\}$ and $\{m = k = 1\}$,

$$D_n^+ v_{0,2j} = +v_{0,2j}(u_{n,2j}^{(n)} - u_{n,2(j-1)}^{(n)}), \quad (106)$$

$$\begin{aligned} D_n^+ v_{1,2j} &= -v_{0,2j}u_{n-1,2(j-1)}^{(n)} - (-1)^n v_{1,2j}(u_{n,2j-1}^{(n)} - u_{n,2j}^{(n)}), \\ D_n^+ v_{1,2j+1} &= +v_{0,2(j+1)}u_{n-1,2j+1}^{(n)} - (-1)^n v_{1,2j+1}(u_{n,2j}^{(n)} - u_{n,2j+1}^{(n)}) \end{aligned} \quad (107)$$

and

$$D_{2n}^- v_{0,2j} = -v_{0,2j}(v_{2n,2j}^{(n)} - v_{2n,2(j-1)}^{(n)}), \quad (108)$$

$$\begin{aligned} D_{2n}^- v_{1,2j} &= v_{0,2j}v_{2n+1,2(j-1)}^{(n)} + v_{1,2j}(v_{2n,2j-1}^{(n)} - v_{2n,2j}^{(n)}), \\ D_{2n}^- v_{1,2j+1} &= -v_{0,2(j+1)}v_{2n+1,2j+1}^{(n)} + v_{1,2j+1}(v_{2n,2j}^{(n)} - v_{2n,2j+1}^{(n)}), \end{aligned} \quad (109)$$

respectively, which involve the two fields, $v_{0,2j}$ and $v_{1,j}$. When deriving these equations we have used the conditions (92) and the relation

$$\sum_{p=0}^k (v_{p,j}^{(n)} v_{k-p,j+p-2n} - v_{k-p,j}^{(n)} v_{p,j+k-p-2}) = 0 \quad (110)$$

at $k = 2n$ and $k = 2n + 1$ which is a direct consequence of the identity $(L^-)^n L^- - L^-(L^-)^n = 0$. Equations (106–109) can further be simplified if one introduces the new basis $\{v_{0,2j}, v_{1,2j}, v_{1,2j-1}\} \Rightarrow \{g_j, F_j, \bar{F}_j\}$, according to the formulae

$$v_{0,2j} = g_{2j}g_{2j-1}, \quad v_{1,2j} = g_{2j}F_j, \quad v_{1,2j-1} = g_{2j-1}\bar{F}_j, \quad (111)$$

where F_j, \bar{F}_j (g_j) are new fermionic (bosonic) fields. Then eqs. (106–109) become

$$D_n^+ \ln g_j = u_{n,j}^{(n)} - u_{n,j-1}^{(n)}, \quad (112)$$

$$D_n^+ F_j = -g_{2j-1} u_{n-1,2(j-1)}^{(n)}, \quad D_n^+ \overline{F}_j = +g_{2j} u_{n-1,2j-1}^{(n)} \quad (113)$$

and

$$D_{2n}^- \ln g_j = -v_{2n,j}^{(n)} + v_{2n,j-1}^{(n)}, \quad (114)$$

$$D_{2n}^- F_j = +g_{2j-1} v_{2n+1,2(j-1)}^{(n)}, \quad D_{2n}^- \overline{F}_j = -g_{2j} v_{2n+1,2j-1}^{(n)}. \quad (115)$$

Now, using eqs. (96) one can resolve eqs. (113) at $n = 1$ and express the field g_j in terms of the fields F_j, \overline{F}_j ,

$$g_{2j-1} = -D_1^+ F_j, \quad g_{2j} = +D_1^+ \overline{F}_j. \quad (116)$$

Finally, eliminating g_j (116) from eqs. (112) and (114) we obtain the following set of equations for the fields F_j, \overline{F}_j :

$$D_n^+ \ln D_1^+ \overline{F}_j = u_{n,2j}^{(n)} - u_{n,2j-1}^{(n)}, \quad D_n^+ \ln D_1^+ F_j = u_{n,2j-1}^{(n)} - u_{n,2(j-1)}^{(n)} \quad (117)$$

and

$$\begin{aligned} D_{2n}^- \ln D_1^+ \overline{F}_j &= -v_{2n,2j}^{(n)} + v_{2n,2j-1}^{(n)}, \\ D_{2n}^- \ln D_1^+ F_j &= -v_{2n,2j-1}^{(n)} + v_{2n,2(j-1)}^{(n)}. \end{aligned} \quad (118)$$

Alternatively, substituting g_j from eqs. (116) into eqs. (113) and (115) we have

$$D_n^+ F_j = (D_1^+ F_j) u_{n-1,2(j-1)}^{(n)}, \quad D_n^+ \overline{F}_j = (D_1^+ \overline{F}_j) u_{n-1,2j-1}^{(n)} \quad (119)$$

and

$$\begin{aligned} D_{2n}^- F_j &= -(D_1^+ F_j) v_{2n+1,2(j-1)}^{(n)}, \\ D_{2n}^- \overline{F}_j &= -(D_1^+ \overline{F}_j) v_{2n+1,2j-1}^{(n)}. \end{aligned} \quad (120)$$

Now, it is necessary to express the functionals $u_{k,j}^{(n)}$ and $v_{k,j}^{(n)}$ entering into the right-hand sides of eqs. (117–118) (or eqs. (119–120)) in terms of the fields $\{F_j, \overline{F}_j\}$ in order to have a closed set of equations for the latter. With this goal in mind, let us consider eqs. (103) and eqs. (104) at $n = 1$,

$$\begin{aligned} D_2^- u_{k,j}^{(m)} &= v_{0,j} u_{k-2,j-2}^{(m)} - v_{0,j-k+m+2} u_{k-2,j}^{(m)} \\ &+ (-1)^m v_{1,j} u_{k-1,j-1}^{(m)} + (-1)^k v_{1,j-k+m+1} u_{k-1,j}^{(m)} \end{aligned} \quad (121)$$

and

$$D_1^+ v_{k,j}^{(m)} - (-1)^k v_{k,j}^{(m)} (u_{1,j} - u_{1,j+k-2m}) = v_{k-1,j+1}^{(m)} + (-1)^k v_{k-1,j}^{(m)}, \quad (122)$$

where $v_{0,j}$ and $v_{1,j}$ should be expressed in terms of $\{F_j, \bar{F}_j\}$ using eqs. (92), (111) and (116),

$$\begin{aligned} v_{0,2j+1} &= 0, & v_{0,2j} &= -(D_1^+ F_j) D_1^+ \bar{F}_j, \\ v_{1,2j} &= F_j D_1^+ \bar{F}_j, & v_{1,2j-1} &= -(D_1^+ F_j) \bar{F}_j. \end{aligned} \quad (123)$$

Substituting

$$u_{1,j} - u_{1,j+k-2m} = +D_1^+ \ln \prod_{n=1}^{2m-k} g_{j+k-2m+n}, \quad 2m > k, \quad (124)$$

obtained from eq. (112) at $n = 1$, into eq. (122) and introducing the new basis $v_{k,j}^{(m)} \Rightarrow \tilde{v}_{k,j}^{(m)}$,

$$v_{2m,j}^{(m)} = \tilde{v}_{2m,j}^{(m)}, \quad v_{k,j}^{(m)} = \tilde{v}_{k,j}^{(m)} \prod_{n=1}^{2m-k} g_{j+k-2m+n}, \quad 2m > k \quad (125)$$

eq. (122) becomes simpler

$$D_1^+ \tilde{v}_{k,j}^{(m)} = g_{j+1} \tilde{v}_{k-1,j+1}^{(m)} + (-1)^k g_{j+k-2m} \tilde{v}_{k-1,j}^{(m)}, \quad 2m \geq k, \quad (126)$$

where g_j is given in terms of $\{F_j, \bar{F}_j\}$ by eqs. (116).

The equations (121) and (126) derived represent recurrent relations connecting the functional $u_{k,j}^{(n)}$ and $\tilde{v}_{k,j}^{(n)}$ with $\{u_{k-1,i}^{(n)}, u_{k-2,i}^{(n)}\}$ and $\tilde{v}_{k-1,i}^{(n)}$, respectively. Being iterated with the starting values $\{u_{-1,j}^{(n)} = 0, u_{0,j}^{(n)} = 1\}$ (96) and $\tilde{v}_{0,j}^{(n)} = 1$, respectively, they allow one to express the functionals $u_{n,j}^{(n)}$ and $v_{2n,j}^{(n)} \equiv \tilde{v}_{2n,j}^{(n)}$ in terms of $\{F_j, \bar{F}_j\}$ after the n -th and $2n$ -th steps of the iteration procedure, respectively. The latter yield the flows D_n^+ and D_{2n}^- of the fields F_j and \bar{F}_j via eqs. (117) and (118).

For illustration, we present explicitly the flows D_n^+ (117) at $n = 1, n = 2$ and $n = 4$ constructed by the above-described algorithmic procedure which allows one to pass step by step,

$$\begin{aligned} D_2^- D_1^+ \ln D_1^+ F_{j+1} &= +F_j D_1^+ \bar{F}_j - F_{j+1} D_1^+ \bar{F}_{j+1}, \\ D_2^- D_1^+ \ln D_1^+ \bar{F}_j &= -(D_1^+ F_j) \bar{F}_j + (D_1^+ F_{j+1}) \bar{F}_{j+1}, \end{aligned} \quad (127)$$

$$D_2^+ F_j = (D_1^+)^2 F_j, \quad D_2^+ \bar{F}_j = (D_1^+)^2 \bar{F}_j, \quad (128)$$

$$\begin{aligned} D_4^+ F_j &= -(D_1^+)^4 F_j - 2(D_1^+ F_j)(D_2^-)^{-1}(D_1^+)^2(F_j D_1^+ \bar{F}_j) \\ &\quad + 2((D_1^+)^2 F_j)(D_2^-)^{-1}(D_1^+)^2(F_j \bar{F}_j), \\ D_4^+ \bar{F}_j &= +(D_1^+)^4 \bar{F}_j + 2(D_1^+ \bar{F}_j)(D_2^-)^{-1}(D_1^+)^2((D_1^+ F_j) \bar{F}_j) \\ &\quad + 2((D_1^+)^2 \bar{F}_j)(D_2^-)^{-1}(D_1^+)^2(F_j \bar{F}_j). \end{aligned} \quad (129)$$

When deriving eqs. (128–129) we have used eqs. (127) in order to express the fields $\{F_{j+i}, \bar{F}_{j+i}\}$, appearing at different lattice points $j+i$, in terms of the fields $\{F_j, \bar{F}_j\}$ at the lattice point j .

Equations (127) reproduce the $N = (1|1)$ superfield form of the $N = (0|2)$ superconformal 2DTL equation [4] which is the minimal supersymmetrization of the 2DTL equation (13). Let us discuss this point in more detail. Thus, in terms of the superfield components

$$V_j \equiv D_1^+ \bar{F}_j|, \quad \bar{\Psi}_j \equiv \bar{F}_j|, \quad U_j \equiv D_1^+ F_j|, \quad \Psi_j \equiv F_j|, \quad (130)$$

where U_j, V_j ($\Psi_j, \bar{\Psi}_j$) are bosonic (fermionic) fields and $|$ means the $t_1^+ \rightarrow 0$ limit, eqs. (127) become

$$\begin{aligned} \partial_2^+ (\partial_2^- \ln(U_j V_{j-1}) - \Psi_j \bar{\Psi}_j + \Psi_{j-1} \bar{\Psi}_{j-1}) &= 0, \\ \partial_2^- (\frac{1}{U_j} \partial_2^+ \Psi_j) &= V_{j-1} \Psi_{j-1} - V_j \Psi_j, \\ \partial_2^- (\frac{1}{V_j} \partial_2^+ \bar{\Psi}_j) &= U_{j+1} \bar{\Psi}_{j+1} - U_j \bar{\Psi}_j, \\ \partial_2^- \partial_2^+ \ln V_j &= U_{j+1} V_{j+1} - U_j V_j + (\partial_2^+ \Psi_{j+1}) \bar{\Psi}_{j+1} - (\partial_2^+ \Psi_j) \bar{\Psi}_j. \end{aligned} \quad (131)$$

The first equation of system (131) has the form of a conservation law with respect to the coordinate t_2^+ . Resolving this equation in the form

$$\partial_2^- \ln(U_j V_{j-1}) - \Psi_j \bar{\Psi}_j + \Psi_{j-1} \bar{\Psi}_{j-1} = \partial_+ \ln(\eta_{j-1}(t_2^-)/\eta_j(t_2^-)) \quad (132)$$

and rescaling the fields

$$u_j := \eta_j U_j, \quad v_j := \frac{V_j}{\eta_j}, \quad \psi_j := \eta_j \Psi_j, \quad \bar{\psi}_j := \frac{\bar{\Psi}_j}{\eta_j} \quad (133)$$

we rewrite equations (131) in an equivalent component

$$\begin{aligned}
\partial_2^- \ln(u_j v_{j-1}) &= \psi_j \bar{\psi}_j - \psi_{j-1} \bar{\psi}_{j-1}, \\
\partial_2^- \left(\frac{1}{u_j} \partial_2^+ \psi_j \right) &= v_{j-1} \psi_{j-1} - v_j \psi_j, \\
\partial_2^- \left(\frac{1}{v_j} \partial_2^+ \bar{\psi}_j \right) &= u_{j+1} \bar{\psi}_{j+1} - u_j \bar{\psi}_j, \\
\partial_2^- \partial_2^+ \ln v_j &= u_{j+1} v_{j+1} - u_j v_j + (\partial_2^+ \psi_{j+1}) \bar{\psi}_{j+1} - (\partial_2^+ \psi_j) \bar{\psi}_j
\end{aligned} \tag{134}$$

and superfield

$$\begin{aligned}
D_2^- \ln \left((D_1^+ F_{j+1}) (D_1^+ \bar{F}) \right) &= -F_j \bar{F}_j + F_{j+1} \bar{F}_{j+1}, \\
D_2^- D_1^+ \ln D_1^+ \bar{F}_j &= -(D_1^+ F_j) \bar{F}_j + (D_1^+ F_{j+1}) \bar{F}_{j+1}
\end{aligned} \tag{135}$$

form where an arbitrary function $\eta_j(t_2^-)$, introduced in eq. (132), completely disappears. The equations (134) reproduce the component form of the $N = (0|2)$ 2DTL equation [4] which can be reduced to the one-dimensional $N = 2$ supersymmetric Toda chain equations [22] by the reduction constraint $\partial_2^+ = \partial_2^-$. Let us also point out that bosonic and fermionic symmetries of this reduction were analyzed in detail in [23]. In the bosonic limit, when all fermionic fields are set to zero, equations (134) become

$$\partial_2^- \ln(u_j v_{j-1}) = 0, \quad \partial_2^- \partial_2^+ \ln v_j = u_{j+1} v_{j+1} - u_j v_j, \tag{136}$$

and the equation, resulting obviously from them, for the function $b_j \equiv -u_j v_j$

$$\partial_2^- \partial_2^+ \ln b_j = -b_{j+1} + 2b_j - b_{j-1} \tag{137}$$

reproduces the 2DTL equation (13).

As concerns eqs. (129), they represent minimal supersymmetrization of the Davey-Stewartson equation [24] which is the $(2+1)$ -dimensional generalization of the $(1+1)$ -dimensional Nonlinear Schroedinger equation.

Let us remark that the $N = (0|2)$ 2DTL equation (127) as well as the equations (128–129) possess the following involution:

$$(F_j)^* = \bar{F}_{i-j}, \quad (\bar{F}_{i-j})^* = F_{i-j}, \tag{138}$$

where $i \in \mathbb{Z}$ is an arbitrary fixed number.

From the algebra (93) we learn that only bosonic flows D_{2n}^\pm of the $N = (0|2)$ 2DTL hierarchy commute simultaneously with the derivatives D_1^+ and D_2^- entering into the $N = (0|2)$ 2DTL equation (127),

$$[D_1^+, D_{2n}^\pm] = [D_2^-, D_{2n}^\pm] = 0, \quad (139)$$

while the fermionic flows D_{2n+1}^\pm do not. Due to this reason the bosonic flows D_{2n}^\pm (117–118) form symmetries of the $N = (0|2)$ 2DTL equation (127), while the fermionic flows D_{2n+1}^\pm do not. Conversely, the $N = (0|2)$ 2DTL equation (127) forms the infinite-dimensional group of the discrete Darboux-Baeklund symmetries for the hierarchy of the bosonic flows D_{2n}^\pm (117–118) (particularly, eqs. (128–129)). In other words, if the set $\{F_j, \bar{F}_j\}$ is a solution of this hierarchy, then the set $\{F_{j+1}, \bar{F}_{j+1}\}$, related to the former by eqs. (127), is a solution of the hierarchy as well.

4.3 Fermionic symmetries of $N = (0|2)$ 2DTL equation

In this subsection we construct fermionic symmetries of the $N = (0|2)$ 2DTL equation (127) and their algebra. This construction is similar to the construction of fermionic symmetries of the STL hierarchy considered in the subsection 3.3. This permits one to present here its main steps in a telegraphic style and refer the reader to the subsection 3.3 for more details.

First, let us consider eqs. (100) at $n = 1$,

$$D_1^+ u_{k,j}^{(2n)} + (-1)^k u_{k,j}^{(2n)} (u_{1,j-k+2n} - u_{1,j}) = u_{k+1,j+1}^{(2n)} + (-1)^k u_{k+1,j}^{(2n)}. \quad (140)$$

Substituting

$$u_{1,j-k+2n} - u_{1,j} = (D_2^-)^{-1} (v_{1,j} + v_{1,j+1} - v_{1,j-k+2n} - v_{1,j-k+2n+1}) \quad (141)$$

derived from eqs. (121), into eqs. (140) the latter become

$$\begin{aligned} & D_1^+ u_{k,j}^{(2n)} + (-1)^k u_{k,j}^{(2n)} (D_2^-)^{-1} (v_{1,j} + v_{1,j+1} - v_{1,j-k+2n} - v_{1,j-k+2n+1}) \\ &= u_{k+1,j+1}^{(2n)} + (-1)^k u_{k+1,j}^{(2n)}, \end{aligned} \quad (142)$$

where $v_{1,j}$ should be expressed in terms of $\{F_j, \overline{F}_j\}$ via eqs. (123). The derived system of equations (121) and (142) for the functionals $u_{k,j}^{(2n)}$ can be treated as the result of the application of the recursive chain of the substitutions (121) to the symmetry equation corresponding to the symmetries D_{2n}^+ (117) of the $N = (0|2)$ 2DTL equation (127). Equivalently, this system represents the consistency condition for the algebra (139) realized on the shell of the $N = (0|2)$ 2DTL equation (127). Therefore, one can construct the relevant, for the problem under consideration, solutions of equations (121) and (142) forgetting about both the way how they were actually derived and their relation to the $N = (0|2)$ hierarchy.

It is a matter of simple direct calculations to verify that eqs. (121) and (142) possess both bosonic

$$u_{0,j}^{(2n)} = 1, \quad u_{l,j}^{(2n)} = 0, \quad l < 0 \quad (143)$$

and fermionic

$$u_{-1,j}^{(2n)} = (-1)^{j+1}\epsilon, \quad u_{l,j}^{(2n)} = 0, \quad l < -1 \quad (144)$$

solutions where ϵ is a dimensionless fermionic constant.

The bosonic solution (143) corresponds to the bosonic symmetries D_{2n}^+ (117) of the $N = (0|2)$ 2DTL equation (127) discussed in the previous subsection (see the paragraph after eq. (126)).

Now, we would like to concentrate on the fermionic solution (144) aiming to elaborate the corresponding fermionic symmetries we are looking for. Let us represent the bosonic time derivative D_{2n}^+ corresponding to the solution (144) and the functionals $u_{k,j}^{(2n)}$ entering into eqs. (117), (121), (144) and (139) in the following form:

$$D_{2n}^+ := \epsilon \mathcal{D}_{2n+1}^+, \quad u_{k,j}^{(2n)} := \epsilon \mathcal{U}_{k+1,j}^{(2n+1)} \quad (145)$$

defining a new fermionic evolution derivative \mathcal{D}_{2n+1}^+ and functionals $\mathcal{U}_{k,j}^{(2n+1)}$. Then the fermionic constant ϵ enters linearly into both sides of eqs. (117), (121), (144) and (139) which now become

$$\begin{aligned} \mathcal{D}_{2n+1}^+ \ln D_1^+ \overline{F}_j &= \mathcal{U}_{2n+1,2j}^{(2n+1)} - \mathcal{U}_{2n+1,2j-1}^{(2n+1)}, \\ \mathcal{D}_{2n+1}^+ \ln D_1^+ F_j &= \mathcal{U}_{2n+1,2j-1}^{(2n+1)} - \mathcal{U}_{2n+1,2(j-1)}^{(2n+1)}, \end{aligned} \quad (146)$$

$$\begin{aligned}
D_2^- \mathcal{U}_{k,j}^{(2n+1)} &= v_{0,j} \mathcal{U}_{k-2,j-2}^{(2n+1)} - v_{0,j-k+2n+3} \mathcal{U}_{k-2,j}^{(2n+1)} \\
&\quad - v_{1,j} \mathcal{U}_{k-1,j-1}^{(2n+1)} + (-1)^k v_{1,j-k+2n+2} \mathcal{U}_{k-1,j}^{(2n+1)}, \quad (147)
\end{aligned}$$

$$\mathcal{U}_{0,j}^{(2n+1)\pm} = (-1)^{j+1}, \quad \mathcal{U}_{-1,j}^{(2n+1)\pm} = 0 \quad (148)$$

and

$$\{D_1^+, \mathcal{D}_{2n+1}^+\} = [D_2^-, \mathcal{D}_{2n+1}^+] = 0, \quad (149)$$

respectively. From these equations we see that the fermionic flows \mathcal{D}_{2m+1}^+ actually do not depend on ϵ and anticommute (commute) with the fermionic (bosonic) derivative D_1^+ (D_2^-) (149) entering into the $N = (0|2)$ 2DTL equation (127); so \mathcal{D}_{2m+1}^\pm form fermionic symmetries of the latter.

Now, let us establish the algebra of the fermionic symmetries \mathcal{D}_{2n+1}^+ (146–148).

First using eqs. (146) and (149) we calculate the fermionic symmetry \mathcal{D}_1^+

$$\mathcal{D}_1^+ F_j = -D_1^+ F_j, \quad \mathcal{D}_1^+ \overline{F}_j = D_1^+ \overline{F}_j \quad (150)$$

and its algebra

$$\{\mathcal{D}_1^+, \mathcal{D}_1^+\} = -\{D_1^+, D_1^+\} \equiv -2D_2^+. \quad (151)$$

Then, we use eqs. (150) in order to replace D_1^+ by \mathcal{D}_1^+ in the expressions for both the bosonic D_{2n}^\pm (117–118) and fermionic \mathcal{D}_{2n+1}^+ (146–148) symmetries transforming them into the new basis

$$\begin{aligned}
\widehat{u}_{k,j}^{(2n+1)} &:= c_k (-1)^{(k+1)(j+1)} \mathcal{U}_{k,j}^{(2n+1)}, \quad \widehat{u}_{k,j}^{(2n)} := c_k (-1)^{kj} u_{k,j}^{(2n)}, \\
\widehat{v}_{k,j}^{(m)} &:= (-1)^k \widetilde{v}_{k,j}^{(m)}, \quad c_{2n} = c_{2n+1} \equiv (-1)^n, \\
\widehat{\mathcal{D}}_{2n+1}^+ &:= c_{2n+1} \mathcal{D}_{2n+1}^+, \quad \widehat{\mathcal{D}}_{2n}^+ := c_{2n} D_{2n}^+, \quad \widehat{\mathcal{D}}_{2n}^- := D_{2n}^- \quad (152)
\end{aligned}$$

which is defined by a single requirement that the form of the symmetries in this basis is as close as possible to the form of the flows D_n^+ and D_{2n}^- (117–118) of the $N = (0|2)$ 2DTL hierarchy whose algebra (93) is known.

In the new basis (152) the symmetries \mathcal{D}_{2n+1}^+ (146–148) and \mathcal{D}_{2n}^\pm (117–118) as well as the algebra (151) become

$$\begin{aligned}
\widehat{\mathcal{D}}_n^+ \ln \widehat{\mathcal{D}}_1^+ \overline{F}_j &= \widehat{u}_{n,2j}^{(n)} - \widehat{u}_{n,2j-1}^{(n)}, \\
\widehat{\mathcal{D}}_n^+ \ln \widehat{\mathcal{D}}_1^+ F_j &= \widehat{u}_{n,2j-1}^{(n)} - \widehat{u}_{n,2(j-1)}^{(n)}, \\
\widehat{\mathcal{D}}_2^- \widehat{u}_{k,j}^{(n)} &= v_{0,j} \widehat{u}_{k-2,j-2}^{(n)} - v_{0,j-k+n+2} \widehat{u}_{k-2,j}^{(n)} \\
&\quad + (-1)^n v_{1,j} \widehat{u}_{k-1,j-1}^{(n)} + (-1)^k v_{1,j-k+n+1} \widehat{u}_{k-1,j}^{(n)}, \\
\widehat{u}_{0,j}^{(n)\pm} &= 1, \quad \widehat{u}_{-1,j}^{(n)\pm} = 0,
\end{aligned} \tag{153}$$

and

$$\begin{aligned}
\widehat{\mathcal{D}}_{2n}^- \ln \widehat{\mathcal{D}}_1^+ \overline{F}_j &= -\widehat{v}_{2n,2j}^{(n)} + \widehat{v}_{2n,2j-1}^{(n)}, \\
\widehat{\mathcal{D}}_{2n}^- \ln \widehat{\mathcal{D}}_1^+ F_j &= -\widehat{v}_{2n,2j-1}^{(n)} + \widehat{v}_{2n,2(j-1)}^{(n)}, \\
\widehat{\mathcal{D}}_1^+ \widehat{v}_{k,j}^{(m)} &= g_{j+1} \widehat{v}_{k-1,j+1}^{(m)} + (-1)^k g_{j+k-2m} \widehat{v}_{k-1,j}^{(m)}, \quad \widehat{v}_{0,j}^{(n)\pm} = 1
\end{aligned} \tag{154}$$

as well as

$$\{\widehat{\mathcal{D}}_1^\mp, \widehat{\mathcal{D}}_1^\mp\} = 2\widehat{\mathcal{D}}_2^\mp, \tag{155}$$

respectively, where

$$\begin{aligned}
g_{2j-1} &= -\widehat{\mathcal{D}}_1^+ F_j, \quad g_{2j} = +\widehat{\mathcal{D}}_1^+ \overline{F}_j, \\
v_{0,2j+1} &= 0, \quad v_{0,2j} = -(\widehat{\mathcal{D}}_1^+ F_j) \widehat{\mathcal{D}}_1^+ \overline{F}_j, \\
v_{1,2j} &= F_j \widehat{\mathcal{D}}_1^+ \overline{F}_j, \quad v_{1,2j-1} = -(\widehat{\mathcal{D}}_1^+ F_j) \overline{F}_j.
\end{aligned} \tag{156}$$

The relations (153–156) completely reproduce the corresponding relations (117–118), (121–122), (151), (116) and (123) for the flows of the $N = (0|2)$ 2DTL hierarchy where, however, the evolution derivatives D_{2n}^\pm and D_{2n+1}^+ are replaced by $\widehat{\mathcal{D}}_{2n}^\pm$ and $\widehat{\mathcal{D}}_{2n+1}^+$, respectively. Therefore, one can conclude that the algebra of the evolution derivatives $\widehat{\mathcal{D}}_{2n}^\pm$ and $\widehat{\mathcal{D}}_{2n+1}^+$ have also to reproduce the algebra of the evolution derivatives D_{2n}^\pm and D_{2n+1}^+ (93). Thus, we are led to the following formulae for both this algebra and the algebras (139), (149) as well as (151) transformed to the basis (152)

$$\begin{aligned}
\{D_1^+, D_2^-\} &= 0, \quad \{D_1^+, D_1^+\} = -2\widehat{\mathcal{D}}_2^+, \\
[D_1^+, \widehat{\mathcal{D}}_{2n}^\pm] &= \{D_1^+, \widehat{\mathcal{D}}_{2n+1}^+\} = [D_2^-, \widehat{\mathcal{D}}_{2n}^\pm] = [D_2^-, \widehat{\mathcal{D}}_{2n+1}^+] = 0, \\
[\widehat{\mathcal{D}}_n^+, \widehat{\mathcal{D}}_{2l}^\pm] &= [\widehat{\mathcal{D}}_{2n}^-, \widehat{\mathcal{D}}_{2l}^\pm] = 0, \quad \{\widehat{\mathcal{D}}_{2n+1}^+, \widehat{\mathcal{D}}_{2l+1}^+\} = 2\widehat{\mathcal{D}}_{2(n+l+1)}^+.
\end{aligned} \tag{157}$$

The symmetries \mathcal{D}_{2n+1}^+ (146–148) of the $N = (2|2)$ 2DTL equation (127) are actually also symmetries of all the bosonic flows D_{2l}^\pm (117–118) of the $N = (2|2)$ 2DTL hierarchy because of the commutation relations

$$[\mathcal{D}_{2n+1}^+, D_{2l}^\pm] = 0 \quad (158)$$

following from the algebra (157) and relations (152).

The existence of the fermionic symmetries \mathcal{D}_{2n+1}^+ (146–148) creates a new interesting problem of constructing both additional evolution equations for the Lax operators L^\pm (91) generated by \mathcal{D}_{2n+1}^+ and commutation relations between the latter and the fermionic flows D_{2n+1}^+ (117) of the $N = (0|2)$ 2DTL hierarchy. We hope to return to this problem in future.

To close this section, let us point out that the $N = (0|2)$ supersymmetry and $N = (0|2)$ superfield formulation of both the $N = (0|2)$ 2DTL equation (127) and the bosonic flows D_{2l}^\pm (117–118) of the $N = (0|2)$ 2DTL hierarchy can easily be uncovered from the approach and formulae of this subsection. In order to see that, it is enough only to introduce a new, $N = 2$ basis $\{D_+, \overline{D}_+\}$ in the space of the fermionic evolution derivatives $\{D_1^+, \mathcal{D}_1^+\}$, namely:

$$D_+ := \frac{1}{2}(\mathcal{D}_1^+ + D_1^+), \quad \overline{D}_+ := \frac{1}{2}(D_1^+ - \mathcal{D}_1^+) \quad (159)$$

which form the algebra of the $N = 2$ supersymmetry

$$D_+^2 = \overline{D}_+^2 = 0, \quad \{D_+, \overline{D}_+\} = \partial_+, \quad (160)$$

where we have introduced the notation $(D_1^\pm)^2 \equiv \partial_2^\pm := \partial_\pm$. Then, relations (150) become

$$D_+ F_j = 0, \quad \overline{D}_+ \overline{F}_j = 0 \quad (161)$$

and have the form of the $N = (0|2)$ chirality constraints for the chiral and antichiral $N = (0|2)$ superfields F_j and \overline{F}_j , respectively. For illustration, we present the $N = (0|2)$ 2DTL equation (135) and the supersymmetric generalization of the Davey-Stewartson equation (129) identically rewritten to this basis

$$\partial_- \ln \left((\overline{D}_+ F_{j+1})(D_+ \overline{F}_j) \right) = -F_j \overline{F}_j + F_{j+1} \overline{F}_{j+1} \quad (162)$$

and

$$\begin{aligned} D_4^+ F_j &= -\partial_+^2 F_j + 2D_+ \left((\overline{D}_+ F_j) \partial_-^{-1} \partial_+ (F_j \overline{F}_j) \right), \\ D_4^+ \overline{F}_j &= +\partial_+^2 \overline{F}_j + 2\overline{D}_+ \left((D_+ \overline{F}_j) \partial_-^{-1} \partial_+ (F_j \overline{F}_j) \right), \end{aligned} \quad (163)$$

respectively which possess both manifest $N = (0|2)$ supersymmetry and $N = (0|2)$ superfield form.

5 Generalizations

In this section, we briefly describe generalizations of the supersymmetric $N = (0|2)$ 2DTL hierarchy discussed in the preceding section.

We propose the following set of the consistent operator equations:

$$\begin{aligned} D_n^+ L^+ &= (-1)^n (((L^+)_*^n)_+)^* L^+ - (L^+)^{*(n)} ((L^+)_*^n)_+ \\ &\quad + (1 - (-1)^n) (L^+)_*^{n+1}, \\ D_n^+ L^- &= ((L^+)_*^n)_+ L^- - (L^-)^{*(n)} ((L^+)_*^n)_+, \\ D_{2Mn}^- L^+ &= (((L^-)^n)_-)^* L^+ - (L^+)((L^-)^n)_-, \\ D_{2Mn}^- L^- &= [((L^-)^n)_-, L^-], \quad n \in \mathbb{N}, \end{aligned} \quad (164)$$

$$L^+ = \sum_{k=0}^{\infty} u_{k,j} e^{(1-k)\partial}, \quad L^- = \sum_{k=0}^{\infty} v_{k,j} e^{(k-2M)\partial}, \quad (165)$$

$$u_{0,j} \equiv 1, \quad v_{0,2Mj+1} \equiv 0, \quad v_{0,2Mj+\alpha} \neq 0, \quad \alpha = 2, 3, \dots, 2M \quad (166)$$

generating the non-abelian algebra of the flows

$$\begin{aligned} [D_n^+, D_{2Ml}^-] &= [D_{2n}^+, D_{2l}^+] = [D_{2Mn}^-, D_{2Ml}^-] = 0, \\ \{D_{2n+1}^+, D_{2l+1}^+\} &= 2D_{2(n+l+1)}^+ \end{aligned} \quad (167)$$

which may be realized in the superspace $\{t_n^+, t_{2Mn}^-\}$

$$D_{2Mn}^\pm = \partial_{2Mn}^\pm, \quad D_{2n+1}^+ = \partial_{2n+1}^+ + \sum_{l=1}^{\infty} t_{2l-1}^+ \partial_{2(k+l)}^+, \quad (168)$$

where $M \in \mathbb{N}$ is a fixed number and t_{2n}^+, t_{2Mn}^- (t_{2n+1}^+) are bosonic (fermionic) evolution times. At $M = 1$ eqs. (164–168) reproduce the Lax pair representation of the $N = (0|2)$ 2DTL hierarchy.

At different values M equations (164–168) generate non-equivalent supersymmetric hierarchies. Nevertheless, any hierarchy with $M = M_1$ can be produced by reduction of the hierarchy with $M = nM_1$ ($n \in \mathbb{N}$) if the latter is provided by the following additional reduction constraints: $v_{0,2M_1j+1} \equiv 0$ which are obviously consistent with the original constraints $v_{0,2nM_1j+1} \equiv 0$ entering into the definition (166) of the latter hierarchy.

A detailed analysis of the generalizations proposed here is under way.

6 Conclusion

In this paper, we have clarified the origin of fermionic and bosonic solutions [1, 2, 3] to the symmetry equations corresponding to the 2DTL and $N = (2|2)$ supersymmetric 2DTL equations and established the algebras of the corresponding symmetries. As a byproduct we have also proved the conjecture, proposed in [16], regarding an $N = (2|2)$ superfield formulation of the STL hierarchy. Then, we have proposed the new, $N = (0|2)$ supersymmetric 2DTL hierarchy. Furthermore, we have constructed bosonic and fermionic symmetries of the $N = (0|2)$ 2DTL equation belonging the hierarchy and their algebra to our knowledge for the first time. We have also discussed an $N = (0|2)$ superfield formulation of the $N = (0|2)$ 2DTL hierarchy. Finally, we have generalized the approach developed for the case of the $N = (0|2)$ 2DTL hierarchy and proposed an infinite class of new supersymmetric Toda type hierarchies.

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